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1987 J. Phys. A: Math. Gen. 20 5831

(http://iopscience.iop.org/0305-4470/20/17/018)

J. Phys. A: Math. Gen. 20 (1987) 5831-5848. Printed in the UK

Pseudo-Euclidean Hurwitz pairs and the Kałuża–Klein theories

Julian Ławrynowicz† and Jakub Rembieliński‡§

† Institute of Mathematics, Polish Academy of Sciences, Łódź Branch, Narutowicza 56, PL-90-136 Łódź, Poland

‡ Institute of Physics, University of Łódź, Nowotki 149/153, PL-90-236 Łódź, Poland

Received 19 January 1987, in final form 22 May 1987

Abstract. Following the geometrical concept of pseudo-Euclidean Hurwitz pairs, we give their systematic classification according to the relationship with real Clifford algebras. Next, a generalisation of the pairs to the curvilinear case is given and it is shown that they provide a convenient framework for the Kałuża-Klein theories.

1. Introduction

During the last few years Kałuża's (1921) idea that the unifying stage of the world should be chosen as a high-dimensional space has gained popularity. One of the reasons is that in the context of the Kałuża-Klein theories the gauge group arises in a natural fashion as isometries of the internal space. Due to the 'gravitational' interaction in a high-dimensional spacetime the supplementary dimensions are somehow curled up to the form of a compact internal space with a very small characteristic length scale comparable with the Planck length. Such a theory looks quite reasonable in the bosonic sector; cf Mecklenburg (1984), Wetterich (1983) and Arefeeva and Volovich (1985). On the other hand, fermions can be treated separately and there is still some freedom in choosing their spinor representations and in forming the field equations. This holds true because bosons and fermions are not treated within the same geometrical framework. Certainly one of the ways of unifying the theories is to postulate a supersymmetry (the Kałuża-Klein supergravity).

In this paper we investigate another possibility of introducing a common framework for bosons and fermions. In order to do this we are going to apply our notion of a pseudo-Euclidean Hurwitz pair (S, V) as some generalisation of the hypercomplex numbers. We give complete classification of the pseudo-Euclidean Hurwitz pairs in terms of Clifford algebras; cf, e.g., Porteous (1981). Next we generalise the notion of holomorphic functions to regular mappings $f\colon S \to V$ so that they satisfy the generalised Fueter equations which resemble some Weyl-like equations. The regular mappings form a spinor representation of the corresponding spin group. A connection with the usual Kałuża-Klein theories is obtained by considering a pseudo-Riemannian generalisation of the pseudo-Euclidean Hurwitz pairs.

Thus, in addition to a new mathematical framework of the Kałuża-Klein theories within the concept of a pseudo-Euclidean Hurwitz pair the paper provides constraints for the possible schemes as well as for the admissible types of their symmetries yielded

§ Supported by CPBP 01.09.

by the corresponding generalised Fueter equations and the associated Dirac fields (§6), including the curved case (§5). Physically, this is a kind of a selection rule for the admissible dimensions of the spacetime in question and for the field pattern. In each particular case the low-energy hadron spectrum may be investigated by the method of harmonic expansion; cf Mecklenburg (1984). An additional selection rule may be given by the requirement of compatibility of the algebraic structure of a Hurwitz pair with the global topological structure of the spacetime fibre bundle. It seems that this last conclusion includes cosmological implications.

2. A geometrical realisation of some Clifford algebras as an extension of hypercomplex numbers

Consider two real vector spaces S and V, equipped with non-degenerate pseudo-Euclidean real scalar products $(\ ,\ )_S$ and $(\ ,\ )_V$ with standard properties. For f, g, $h \in V$; a, b, $c \in S$, and α, $\beta \in \mathbb{R}$ we suppose that

$$\begin{array}{lll} (a, b)_S \in \mathbb{R} & (f, g)_V \in \mathbb{R} & \\ (b, a)_S = (a, b)_S & (g, f)_V = \delta(f, g)_V & \delta = 1 \text{ or } -1 \\ (\alpha a, b)_S = \alpha(a, b)_S & (\alpha f, g)_V = \alpha(f, g)_V & \\ (a, b+c)_S = (a, b)_S + (a, c)_S & (f, g+h)_V = (f, g)_V + (f, h)_V. & \end{array} \tag{1}$$

In S and V we choose some bases (ε_α) and (e_j), respectively, with $\alpha = 1, \ldots, \dim S = p$; $k = 1, \ldots, \dim V = n$. We assume that $p \leqslant n$. For

$$\eta \equiv [\eta_{\alpha\beta}] := [(\varepsilon_\alpha, \varepsilon_\beta)_S] \qquad \kappa \equiv [\kappa_{jk}] := [(e_j, e_k)_V] \tag{2}$$

by relations (1), we get immediately

$$\begin{array}{llll} \eta^{-1} \equiv [\eta^{\alpha\beta}] & \eta^T = \eta & \kappa^{-1} \equiv [\kappa^{jk}] & \kappa^T = \delta\kappa \\ \det \eta \neq 0 & \det \kappa \neq 0. & & \end{array}$$

Now, without any loss of generality, we can choose the basis (ε_α) so that

$$\eta = \mathrm{diag}\underbrace{(1, \ldots, 1, -1, \ldots, -1)}_{p \text{ times}} \tag{3}$$

and hence $\eta^{-1} = \eta$. The vectors from S and V have the form

$$a = a^\alpha \varepsilon_\alpha \in S \qquad a^\alpha \in \mathbb{R}$$

and

$$f = f^j e_j \in V \qquad f^k \in \mathbb{R}$$

respectively. With the help of the metric tensors η and κ we can pass from covariant quantities to the contravariant ones and conversely:

$$\begin{array}{llll} a_\alpha = \eta_{\alpha\beta} a^\beta & \varepsilon^\alpha = \eta^{\alpha\beta}\varepsilon_\alpha & f_j = \kappa_{jk} f^k & e^j = \kappa^{jk} e_k \\ a^\alpha = \eta^{\alpha\beta} a_\beta & \varepsilon_\alpha = \eta_{\alpha\beta}\varepsilon^\alpha & f^j = \kappa^{jk} f_k & e_j = \kappa_{jk} e_k. \end{array}$$

Obviously, for f, $g \in V$ and a, $b \in S$ we have

$$(a, b) = a^\alpha b_\alpha = a^\alpha b^\beta \eta_{\alpha\beta} \qquad (f, g) = f^j g_j = f^j g^k \kappa_{jk}.$$

The *multiplication* of elements of S by elements of V is defined as a mapping $S\times V\to V$ with the properties

(i) $(a+b)f=af+bf$ and $a(f+g)=af+ag$ for $f, g\in V$ and $a, b\in S$

(ii) $(a,a)_S(f,g)_V=(af,ag)_V$ (*the generalised Hurwitz condition*)

(iii) there exists the unit element ε_0 in S with respect to the multiplication: $\varepsilon_0 f=f$ for $f\in V$.

The $\mathbb{R}$-linearity of the multiplication follows from (i): we have $\alpha(af)=a(\alpha f)$ for $\alpha\in\mathbb{R}$. By (iii), the multiplication of vectors of V by a real number α is identified with the multiplication by $\alpha\varepsilon_0$.

The product af is uniquely determined by the multiplication scheme for base vectors:

$$\varepsilon_\alpha e_j=c^k_{j\alpha}e_k \qquad \alpha=1,\ldots,p \qquad j,k=1,\ldots,n. \tag{4}$$

The above scheme, together with the postulates (1), yields the following formulae for the real structure constants $c^k_{j\alpha}$:

$$c^k_{j\alpha}=(e^k,\varepsilon_\alpha e_j)_V$$

i.e. they are simply the matrix elements of ε_α treated as an $\mathbb{R}$-linear endomorphism of V; Adem (1975, 1978, 1980).

Hereafter we shall require the *irreducibility* of the multiplication $S\times V\to V$, which means that it does not leave invariant proper subspaces of V. In such a case we shall call (V,S) a *pseudo-Euclidean Hurwitz pair.*

In order to investigate the consequences of the most important condition (ii), let us rewrite it in the coordinate form:

$$\begin{aligned}(af,ag)_V&\equiv\tfrac12 a^\alpha a^\beta f^j f^k[(\varepsilon_\alpha e_j,\varepsilon_\beta e_k)_V+(\varepsilon_\beta e_j,\varepsilon_\alpha e_k)_V]\\&\equiv\tfrac12 a^\alpha a^\beta f^j f^k[(e^r,\varepsilon_\alpha e_j)_V\kappa_{rs}(e^s,\varepsilon_\beta e_k)_V+(e^r,\varepsilon_\beta e_j)_V\kappa_{rs}(e^s,\varepsilon_\alpha e_k)V]\\&=(a,a)_S(f,f)_V\equiv a^\alpha a^\beta f^j f^k\eta_{\alpha\beta}\kappa_{jk}.\end{aligned}$$

Hence

$$c^r_{j\alpha}\kappa_{rs}c^s_{k\beta}+c^r_{j\beta}\kappa_{rs}c^s_{k\alpha}=2\eta_{\alpha\beta}\kappa_{jk}$$

or, equivalently, if I_n stands for the identity $n\times n$ matrix,

$$C_\alpha\bar C_\beta+C_\beta\bar C_\alpha=2\eta_{\alpha\beta}I_n \qquad \alpha,\beta=1,\ldots,\dim S \tag{5}$$

in the matrix notation

$$C_\alpha:=[c^k_{j\alpha}] \qquad \bar C_\alpha:=\kappa C^T_\alpha\kappa^{-1}.$$

It can easily be seen that the $\mathbb{R}$-linearity of ε_α together with relations (4) and (5) are equivalent to the conditions (i) and (ii). Besides, (5) yields the invertibility of C_α. When setting

$$\begin{aligned}&C_\alpha=\mathrm{i}\gamma_\alpha C_t \qquad && t \text{ fixed}\\&\alpha=1,\ldots,p && \alpha\neq t\end{aligned} \tag{6}$$

where i denotes the imaginary unit, we arrive at the following system equivalent to (5):

$$\begin{aligned}&C_t\bar C_t=\eta_{tt}I_n && t\text{ fixed}\\&\bar\gamma_\alpha=-\gamma_\alpha && \operatorname{Re}\gamma_\alpha=0\\&\alpha=1,\ldots,p && \alpha\neq t\\&\{\gamma_\alpha,\gamma_\beta\}\equiv\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2\hat\eta_{\alpha\beta}I_n \qquad && \alpha,\beta=1,\ldots,p \qquad \alpha,\beta\neq t\end{aligned} \tag{7}$$

where

$$\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}/\eta_{tt} \tag{8}$$

$\eta_{\alpha\beta}$ being chosen diagonal as in (3). Clearly,

$$\eta_{tt} = 1 \text{ or } -1. \tag{9}$$

From (7) it follows that γ_α *are generators of a real Clifford algebra* $C^{(r,s)}$ *with* (r, s) *determined by the signature of* $\hat{\eta} := [\hat{\eta}_{\alpha\beta}]$ *and by*

$$r + s = p - 1. \tag{10}$$

The matrices γ_α are chosen in the (imaginary) Majorana representation. Conversely, *any pseudo-Euclidean Hurwitz pair* (V, S) *is a geometrical realisation of a real Clifford algebra* $C^{(r,s)}$, and the relationship is given by the conditions (6)-(9), (r, s) being determined by the signature of $\hat{\eta}$ and by (10).

The above statement does not determine (r, s) uniquely. The precise result requires a deeper mathematical reasoning, given in a separate paper by Ławrynowicz *et al* (1988).

3. Classification of pseudo-Euclidean Hurwitz pairs

Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote the real, complex and quaternion number fields, and let $\mathbb{M}(N, \mathbb{F})$ be the algebra of $N \times N$ matrices over $\mathbb{F}$. In Ławrynowicz and Rembieliński (1986) we have given a classification table of $C^{(r,s)}$ modulo 8 in terms of $\mathbb{M}(N, \mathbb{F})$, based on the famous periodicity theorem for real Clifford algebras; cf Atiyah *et al* (1964). We have also had to take into account the recurrence relations

$$C^{(r,s)} \times C^{(1,1)} = C^{(r+1,s+1)} \qquad C^{(r,s)} \times C^{(2,0)} = C^{(r+2,s)}$$
$$C^{(r,s)} \times C^{(0,2)} = C^{(r,s+2)}.$$

Here, instead of reproducing this table, we are going to consider in detail each of the eight possible cases separately, distinguishing subsequently several particular cases.

3.1. $r - s \equiv 0 \pmod 8$

In this case we have $C^{(r,s)} \sim \mathbb{M}(2^{m/2}, \mathbb{R})$, $m = r + s$, the dimension of the representation space is 2^l, $l = [\frac{1}{2}m]$, where [ ] stands for the function '*entier*'. Since the conditions $\bar{\gamma}_\alpha = -\gamma_\alpha$, re $\gamma_\alpha = 0$ in (7) are satisfied, then $n = 2^l$. We can construct both the real and the imaginary Majorana representation. Now, considering the existence of the matrix $\kappa := [(e_j, e_k)_V]$, we have to introduce, as in Ławrynowicz and Rembieliński (1985b), the sequence

$$\check{\gamma}_\alpha = \gamma_\alpha \qquad \alpha = 1, \ldots, r \qquad \hat{\gamma}_\beta = \gamma_{r+\beta} \qquad \beta = 1, \ldots, s. \tag{11}$$

In turn we can construct the real matrices

$$A = (-i)^r \check{\gamma}_1 \check{\gamma}_2 \ldots \check{\gamma}_r \qquad B = (-i)^s \hat{\gamma}_1 \hat{\gamma}_2 \ldots \hat{\gamma}_s \tag{12}$$

if $s = 0$ we set $B = I_n$. We verify directly their properties:

$$A^T = (-1)^{r(r+1)/2} A \qquad B^T = (-1)^{s(s-1)/2} B \tag{13}$$

and

$$\begin{aligned} A\check{\gamma}_\alpha &= (-1)^{r-1} \check{\gamma}_\alpha A \qquad & B\check{\gamma}_\alpha &= (-1)^s \check{\gamma}_\alpha B \\ A\hat{\gamma}_\beta &= (-1)^r \hat{\gamma}_\beta A \qquad & B\hat{\gamma}_\beta &= (-1)^{s-1} \hat{\gamma}_\beta B. \end{aligned} \tag{14}$$

The formulae (11)-(14) are clearly independent of the case in question; they remain valid also thereafter, in the cases 3.2-3.8. Returning to our particular situation in 3.1 we observe that the matrix κ, if it exists, has to be an element of the Clifford algebra under consideration, namely

$$\kappa = aI_n + \mathrm{i}b^{\alpha}\gamma_{\alpha} + c^{\alpha\beta}\gamma_{\alpha}\gamma_{\beta} + \mathrm{i}d^{\alpha\beta\delta}\gamma_{\alpha}\gamma_{\beta}\gamma_{\delta} + \ldots \tag{15}$$

where the coefficients a, b^{α}, $c^{\alpha\beta}$, $d^{\alpha\beta\delta}, \ldots$, are real and antisymmetric with respect to the transposition of the indices $\alpha, \beta, \delta, \ldots$.

Now let us recall that by the existence of the matrix κ we mean the existence of κ satisfying the constraints of the problem, that is the conditions $\bar{\gamma}_{\alpha} = -\gamma_{\alpha}$, re $\gamma_{\alpha} = 0$ in (7). In other words, we have to find all the particular cases where the expression (15) is consistent with the conditions

$$\kappa\gamma_{\alpha}^{T}\kappa^{-1} = -\gamma_{\alpha} \qquad \alpha = 1, \ldots, p-1$$

or, equivalently,

$$\kappa\check{\gamma}_{\alpha} = \check{\gamma}_{\alpha}\kappa \qquad \alpha = 1, \ldots, r;\ \kappa\hat{\gamma}_{\beta} = -\hat{\gamma}_{\beta}\kappa \qquad \beta = 1, \ldots, s. \tag{16}$$

Of course, the procedure described remains valid also thereafter, in the cases 3.2-3.8, including the necessity and sufficiency of checking the system (16). In our particular situation we arrive at the following conclusion.

In the case $r - s \equiv 0 \pmod 8$ the only possible pseudo-Euclidean Hurwitz pairs are those satisfying one of the following four sets of conditions:

$r = 4(k+k_0)$	$r = 4(k+k_0)+3$	$r = 4(k+k_0)+2$	$r = 4(k+k_0)+1$
$s = 4(k-k_0)$	$s = 4(k+k_0)+3$	$s = 4(k-k_0)+2$	$s = 4(k-k_0)+1$
$\kappa = B \Rightarrow \kappa^{T} = \kappa$	$\kappa = A \Rightarrow \kappa^{T} = \kappa$	$\kappa = B \Rightarrow \kappa^{T} = -\kappa$	$\kappa = A \Rightarrow \kappa^{T} = -\kappa$

where $\Rightarrow$ abbreviates 'which implies', k and k_0 are integers, and $k \geqslant 0$. Of course we assume, as everywhere in this paper, that $r \geqslant 0$ and $s \geqslant 0$ as well. The first particular case with $k = k_0$ is an example of a Euclidean Hurwitz pair, having already been considered in Ławrynowicz and Rembieliński (1985a, b); cf also Rembieliński (1980a, b, 1981). Some other examples appear within each of the further cases 3.2-3.8.

3.2. $r - s \equiv 6 \pmod 8$

We turn our attention to this case, changing the naturally expected order of cases, because of methodological reasons: the case in question is the most analogous to the previous one. Our further programme is shown in figure 1.

We have $C^{(r,s)} \sim \mathbb{M}(2^{(m-2)/2}, \mathbb{H})$ and the dimension of the representation space is $2 \times 2^{[(m-1)^{1/2}]} = 2^{l}$, the notation m, l, and [ ] being as in the case 3.1. The additional factor 2 in the latter equality comes from the fact that $\mathbb{H}$ is regarded as a subalgebra of $\mathbb{M}(2, \mathbb{C})$. All the conditions (7) are satisfied, so $n = 2^{l}$, as before. We can construct the imaginary Majorana representation; its real analogue can only be constructed after doubling the dimension of the representation space. In analogy to the preceding case we conclude that the only four possibilities are the following:

$$r = 4(k+k_0)+6 \qquad r = 4(k+k_0)+7 \qquad r = 4(k+k_0) \qquad r = 4(k+k_0)+1$$

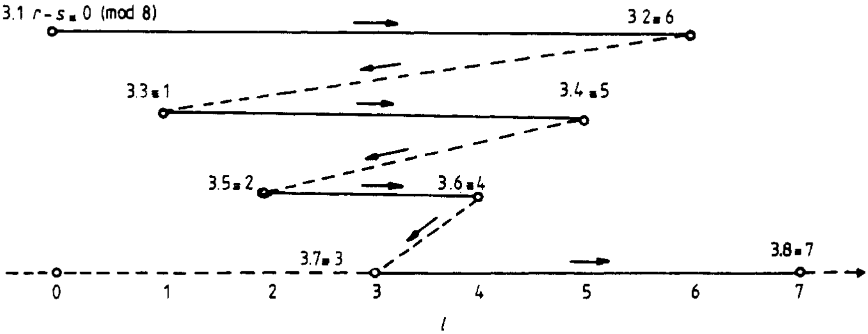

Figure 1. The chosen order of considering the cases $r-s \equiv l \pmod 8$.

$s=4(k-k_0)$	$s=4(k-k_0)+1$	$s=4(k-k_0)+2$	$s=4(k-k_0)+3$
$\kappa = B \Rightarrow \kappa^T = \kappa$	$\kappa = A \Rightarrow \kappa^T = \kappa$	$\kappa = B \Rightarrow \kappa^T = -\kappa$	$\kappa = A \Rightarrow \kappa^T = -\kappa$

where k and k_0 are integers, and $k \geqslant 0$.

3.3. $r-s=1 \pmod 8$

The algebra is the direct sum of two matrix algebras:

$$C^{(r,s)} \sim \mathbb{M}(2^{[m/2]}, \mathbb{R}) + \mathbb{M}(2^{[m/2]}, \mathbb{R}) \tag{17}$$

and the dimension of the representation space is 2^{l+1}. All the conditions (7) are satisfied, so $n=2^{l+1}$. Exactly as in the case 3.1, we can construct now the real as well as the imaginary Majorana representation. Because of (17), we cannot apply the same argument in order to distinguish explicitly all possible Hurwitz pairs. The necessary modification is taking into account that each irreducible representation of $C^{(r,s)}$ can be in our case, owing to the congruence $r-(s+1)\equiv 0 \pmod 8$, embedded in an irreducible representation of the Clifford algebra $C^{(r,s+1)}$ which is isomorphic to the corresponding irreducible matrix ring. Consequently, κ has to belong to $C^{(r,s+1)}$ and this is why we are led to the result that the only four possibilities are the following:

(i)	$r=4(k+k_0)+1$	$\kappa = B \Rightarrow \kappa^T = \kappa$ or
	$s=4(k-k_0)$	$\kappa = A \Rightarrow \kappa^T = -\kappa$
(ii)	$r=4(k+k_0)+3$	$\kappa = A \Rightarrow \kappa^T = \kappa$ or
	$s=4(k-k_0)+2$	$\kappa = B \Rightarrow \kappa^T = -\kappa$
(iii)	$r=4(k+k_0)+4$	$\kappa = \mathrm{i}A\hat{\gamma}_{s+1}$ or
	$s=4(k-k_0)+3$	$\kappa = \mathrm{i}B\hat{\gamma}_{s+1} \Rightarrow \kappa^T = \kappa$
(iv)	$r=4(k+k_0)+2$	$\kappa = \mathrm{i}A\hat{\gamma}_{s+1}$ or
	$s=4(k-k_0)+1$	$\kappa = \mathrm{i}B\hat{\gamma}_{s+1} \Rightarrow \kappa^T = -\kappa$

where k and k_0 are integers, and $k \geqslant 0$.

3.4. $r-s\equiv 5 \pmod 8$

Similarly as in the preceding case, the algebra is the direct sum of two matrix algebras:

$$C^{(r,s)}\sim \mathbb{M}(2^{[(m-2)/2]},\mathbb{H})+\mathbb{M}(2^{[(m-2)/2]},\mathbb{H}) \tag{18}$$

and the dimension of the representation space is $2\times 2^{[(m-2)/2]+1}=2^{l+1}$. The additional factor 2 in the latter equality comes from the fact that $\mathbb{H}$ is regarded as a subalgebra of $\mathbb{M}(2,\mathbb{C})$. All the conditions (7) are satisfied, so $n=2^{l+1}$. We can construct the imaginary Majorana representation; its real analogue can only be constructed after doubling the dimension of the representation space. Exactly as in the case 3.3, by (18), we observe that each irreducible representation of $C^{(r,s)}$ can be embedded in an irreducible representation of $C^{(r+1,s)}$, isomorphic to the corresponding matrix ring. Consequently, κ has to belong to $C^{(r+1,s)}$, so the only four possibilities are the following:

$$\begin{array}{lll}
\text{(i)} & r=4(k+k_0)+7 & \kappa=A\Rightarrow\kappa^T=\kappa \quad \text{or} \\
 & s=4(k-k_0)+2 & \kappa=B\Rightarrow\kappa^T=-\kappa \\
\text{(ii)} & r=4(k+k_0)+5 & \kappa=B\Rightarrow\kappa^T=\kappa \quad \text{or} \\
 & s=4(k-k_0) & \kappa=A\Rightarrow\kappa^T=-\kappa \\
\text{(iii)} & r=4(k+k_0)+6 & \kappa=\mathrm{i}A\check{\gamma}_{r+1} \quad \text{or} \\
 & s=4(k-k_0)+1 & \kappa=\mathrm{i}B\check{\gamma}_{r+1}\Rightarrow\kappa^T=\kappa \\
\text{(iv)} & r=4(k+k_0) & \kappa=\mathrm{i}A\check{\gamma}_{r+1} \quad \text{or} \\
 & s=4(k-k_0)+3 & \kappa=\mathrm{i}B\check{\gamma}_{r+1}\Rightarrow\kappa^T=-\kappa
\end{array}$$

where k and k_0 are integers, and $k\geqslant 0$.

3.5. $r-s\equiv 2 \pmod 8$

In this case $C^{(r,s)}\sim\mathbb{M}(2^{m/2},\mathbb{R})$ and the dimension of the representation space is 2^l. Not all the conditions (7) are satisfied; in order to arrive at the relations $\bar{\gamma}_\alpha=-\gamma_\alpha$, re $\gamma_\alpha=0$ we have to double the dimension 2^l: $n=2\times 2^l=2^{l+1}$ by taking the direct sum of two irreducible copies of the corresponding Clifford algebra $C^{(r,s)}$. In contrast to the case 3.4, we can now construct the real Majorana representation; its imaginary analogue can only be constructed after doubling the dimension of the representation space. By analysing the possibility of the construction of the matrix κ in $C^{(r,s)}$ as well as by embedding its irreducible representations in irreducible representations of $C^{(r+1,s)}$, and then of $C^{(r,s+2)}$, we conclude that the only four possibilities are the following:

$$\begin{array}{lll}
\text{(i)} & r=4(k+k_0)+5 & \kappa=\mathrm{i}B\hat{\gamma}_{s+1}\Rightarrow\kappa^T=\kappa \quad \text{or} \\
 & s=4(k-k_0)+3 & \kappa=A\Rightarrow\kappa^T=-\kappa \\
\text{(ii)} & r=4(k+k_0)+2 & \kappa=B\Rightarrow\kappa^T=\kappa \quad \text{or} \\
 & s=4(k-k_0) & \kappa=B\hat{\gamma}_{s+1}\hat{\gamma}_{s+2}\Rightarrow\kappa^T=-\kappa \\
\text{(iii)} & r=4(k+k_0)+3 & \kappa=A\Rightarrow\kappa^T=\kappa \quad \text{or} \\
 & s=4(k-k_0)+1 & \kappa=\mathrm{i}B\hat{\gamma}_{s+1}\Rightarrow\kappa^T=-\kappa \\
\text{(iv)} & r=4(k+k_0)+4 & \kappa=B\hat{\gamma}_{s+1}\hat{\gamma}_{s+2}\Rightarrow\kappa^T=\kappa \quad \text{or} \\
 & s=4(k-k_0)+2 & \kappa=B\Rightarrow\kappa^T=-\kappa
\end{array} \tag{19}$$

where k and k_0 are integers, and $k\geqslant 0$.

The particular case (19) with $n=8$ and $k=k_0=0$, that is $(n, r, s)=(8, 3, 1)$, is of special interest to us because it provides an interpretation in the five-dimensional Kałuża-Klein theory; note that, by (10), dim $V\equiv p=5$. This case, together with a dual case distinguished below (in the case 3.6), will be treated in detail in § 5.

3.6. $r-s=4 \pmod 8$

We have $C^{(r,s)}\approx \mathbb{M}(2^{(m-2)/2}, \mathbb{H})$ and the dimension of the representation space is $2\times 2^{[(m-2)/2]}=2^l$. The additional factor 2 in the latter equality comes from the fact that $\mathbb{H}$ is regarded as a subalgebra of $\mathbb{M}(2, C)$. Not all the conditions (7) are satisfied; as in the preceding case we have to double the dimension $2^l : n=2^{l+1}$ by taking the direct sum of two irreducible copies of $C^{(r,s)}$. The real and imaginary Majorana representations can only be constructed after doubling the dimension of the representation space. Similarly as in the case 3.5 we conclude that the only four possibilities are the following:

$$
\begin{array}{llll}
\text{(i)} & r=4(k+k_0)+6 & \kappa=\mathrm{i}A\gamma_{r+1}\Rightarrow\kappa^T=\kappa \quad \text{or} & \\
 & s=4(k-k_0)+2 & \kappa=B\Rightarrow\kappa^T=-\kappa & \\
\text{(ii)} & r=4(k+k_0)+7 & \kappa=A\Rightarrow\kappa^T=\kappa \quad \text{or} & \\
 & s=4(k-k_0)+3 & \kappa=A\gamma_{r+1}\gamma_{r+2}\Rightarrow\kappa^T=-\kappa & \\
\text{(iii)} & r=4(k+k_0)+4 & \kappa=B\Rightarrow\kappa^T=\kappa \quad \text{or} & \\
 & s=4(k-k_0) & \kappa=\mathrm{i}A\gamma_{r+1}\Rightarrow\kappa^T=-\kappa & \\
\text{(iv)} & r=4(k+k_0)+5 & \kappa=A\gamma_{r+1}\gamma_{r+2}\Rightarrow\kappa^T=\kappa \quad \text{or} & \\
 & s=4(k+k_0)+1 & \kappa=A\Rightarrow\kappa^T=-\kappa &
\end{array}
\tag{20}
$$

where k and k_0 are integers, and $k\geqslant 0$.

The particular case (20) with $n=8$, $k=0$, and $k_0=-1$, that is $(n, r, s)=(8, 0, 4)$, is of special interest to us because it provides an interpretation in the five-dimensional Kałuża-Klein theory in addition to the already distinguished particular case (19) with $n=8$ and $k=k_0=0$, that is $(n, r, s)=(8, 3, 1)$. Moreover, the same particular case (20), but with $n=8$ and $k=k_0=0$, that is $(n, r, s)=(8, 4, 0)$, is an example of a Euclidean Hurwitz pair. Of course (20) with $k=k_0>0$, but without the restriction $n=8$, is still Euclidean.

3.7. $r-s\equiv 3 \pmod 8$

In this case $C^{(r,s)}\sim\mathbb{M}(2^{[m/2]}, \mathbb{C})$ and the dimension of the representation space is 2^l. Not all the conditions (7) are satisfied; as in the cases 3.5 and 3.6 we have to double the dimension $2^l : n=2^{l+1}$ by taking the direct sum of two irreducible copies of $C^{(r,s)}$. In analogy to the preceding case, the real and imaginary Majorana representations can only be constructed after doubling the dimension of the representation space. Since in this case we choose $C^{(r,s)}$ irreducible and it is isomorphic to the matrix algebra $\mathbb{M}(2^{[m/2]}, \mathbb{C})$, the only possible pseudo-Euclidean Hurwitz pairs are those satisfying one of the following two sets of conditions:

$$
\begin{array}{lll}
\text{(i)} & r=4(k+k_0)+3 & \kappa=A \quad \text{or} \\
 & s=4(k+k_0) & \kappa=B=\kappa^T=\kappa \\
\text{(ii)} & r=4(k+k_0)+5 & \kappa=A \quad \text{or} \\
 & s=4(k-k_0)+2 & \kappa=B=\kappa^T=-\kappa
\end{array}
$$

where k and k_0 are integers, and $k\geqslant 0$.

3.8. $r-s\equiv 7 \pmod 8$

We have $C^{(r,s)}\sim \mathbb{M}(2^{[m/2]},\mathbb{C})$ and the dimension of the representation space is 2^l, exactly as in the case 3.7. In contrast to that case, all the conditions (7) are now satisfied, so $n=2^l$. We can construct the imaginary Majorana representation; its real analogue can only be constructed after doubling the dimension of the representation space. Arguing exactly as in the preceding case, we can see that the only two possibilities are the following:

$$\begin{array}{llll} \text{(i)} & r=4(k+k_0)+1 & \kappa=A \quad \text{or} & \\ & s=4(k-k_0)+2 & \kappa=B\Rightarrow\kappa^T=-\kappa & \\ \text{(ii)} & r=4(k+k_0)+7 & \kappa=A \quad \text{or} & \\ & s=4(k-k_0) & \kappa=B\Rightarrow\kappa^T=\kappa & \end{array} \tag{21}$$

where k and k_0 are integers, and $k\geq 0$. The particular case (19) with $n=8$ and $k=k_0=0$, that is $(n,r,s)=(8,7,0)$, determines the well known octonion algebra. A detailed study of this algebra from the viewpoint of Hurwitz pairs has been given by Kanemaki (1986) recently.

4. Realisation of the Kałuża–Klein ideas within the concept of a Hurwitz pair

In § 2 we have distinguished four Hurwitz pairs providing an interpretation in the five-dimensional Kałuża-Klein theory:

$$\begin{array}{ll} (n,r,s)=(8,3,1) & \kappa=A\Rightarrow\kappa^T=\kappa \\ (n,r,s)=(8,3,1) & \kappa=\mathrm{i}B\hat{\gamma}_{s+1}\Rightarrow\kappa^T=-\kappa \\ (n,r,s)=(8,0,4) & \kappa=B\Rightarrow\kappa^T=\kappa \\ (n,r,s)=(8,0,4) & \kappa=\mathrm{i}A\check{\gamma}_{r+1}\Rightarrow\kappa^T=-\kappa. \end{array} \tag{22}$$

Going further into the problem, we have to take into account the ideas of Kałuża (1921) and Klein (1926) in their contemporary form of the last decade (cf e.g. Lee 1984).

Following the general spirit of these ideas we admit the following postulates.

(i) The spacetime is of dimension $p>4$.

(ii) The spacetime contains a $(p-4)$-dimensional subspace whose compactification is connected with a spontaneous breaking of the symmetry of the vacuum.

(iii) After the compactification the original spacetime has to be replaced by a fibre bundle with a four-dimensional base space $\mathbb{M}_0$ of index 1 or 3 and a $(p-4)$-dimensional compact typical fibre $\mathbb{M}_{\#}$. In general $\mathbb{M}_{\#}$ is the quotient space of a Lie group G and its subgroup G_0. The fibre $\mathbb{M}_{\#}$ is often supposed to be a symmetric space.

(iv) The vacuum solutions corresponding to the fibre bundle $\boldsymbol{B}_{\mathbb{M}}$ with the fibre space $\mathbb{M}$, generated by $\mathbb{M}_0$ and $\mathbb{M}_{\#}$, are obtained from the equations of the gravitational field with the energy-momentum tensor zero. Thus, if $x=(x^\mu;\ \mu=0,\ldots,3)$ and $y=(y^j;\ j=1,\ldots,p-4)$ are any local coordinates in $\mathbb{M}_0$ and $\mathbb{M}_{\#}$, respectively, while $z=(z^a;\ a=1,\ldots,p)$ denotes the corresponding coordinate system in $\mathbb{M}$, the pseudo-Riemannian tensor of $\boldsymbol{B}_{\mathbb{M}}$ has, in the vacuum case, the form

$$[g^0_{ab}(z)]=\begin{bmatrix}[g^0_{\mu\nu}(x)] & 0\\ 0^T & [g^0_{jk}(y)]\end{bmatrix} \tag{23}$$

0 being the zero $4\times(p-4)$ matrix.

(v) *The Kałuża-Klein ansatz* (set-up). For the low energy theory (zero modes) the pseudo-Riemannian tensor of $\boldsymbol{B}_{\mathbb{M}}$ has the form

$$[g_{ab}(z)]=\begin{bmatrix}[g^0_{\mu\nu}(x)+e^2A_\mu A_\nu] & [eA^\alpha_\mu(x)K^j_\alpha(y)g^0_{jk}(y)]\\ [eA^\alpha_\nu(x)K^k_\alpha(y)g^0_{jk}(y)] & [g^0_{jk}(y)]\end{bmatrix} \tag{24}$$

where $A^\alpha_\mu(x)$ are gauge fields, $e=e_0/hc$, e_0 is the electric charge,

$$A_\mu A_\nu := A^\alpha_\mu(x)A^\beta_\nu(x)K^j_\alpha(y)K^k_\beta(y)g^0_{jk}(y)$$

and $K^j_\alpha(y)$ are the Killing vectors connected with the transformations $y^j\to y^j+\varepsilon^\alpha(x)K^j_\alpha(y)$ of the group G treated as the isometry group of $\mathbb{M}_\#$.

In this paper we have to add the requirement for every point z of the fibre bundle $\boldsymbol{B}_M$ of the space S, tangent to M and associated with the vector space V, to form a Hurwitz pair (V,S). The equipment of the bundle $\boldsymbol{B}_M$ with such a structure is considered in § 5. Here, on the basis of the results of § 3, we are only going to classify the possible Kałuża-Klein theories in the above sense.

Hereafter the parameters k and k_0 are integers, and $k\geqslant 0$; we exclude the case $(u,r,s)=(1,0,0)$.

4.1. *Hyperbolic theories with s = 1*

By cases 3.2 and 3.5, and especially (19), we have

$$r=8k+7 \qquad \text{or} \qquad 8k+3 \qquad \kappa=A. \tag{+1}$$

The cases are *hyperbolic* in the sense that, by choosing a suitable basis (e_j) of V, in each case the metric κ of V can be chosen as

$$\kappa=H_n:=\begin{bmatrix}I_{n/2} & 0\\ 0 & -I_{n/2}\end{bmatrix} \tag{25}$$

where $I_{n/2}$ stands for the identity $\frac{1}{2}n\times\frac{1}{2}n$ matrix, which can be checked by a direct calculation. For $k=0$ in the second case of (+1) we arrive at the five-dimensional hyperbolic Kałuża-Klein theory with $(n,r,s)=(8,3,1)$. Similarly, by cases 3.4 and 3.6 we have

$$r=8k+6 \qquad \kappa=\mathrm{i}A\check{\gamma}_{r+1} \tag{+2}$$

$$r=8k+6 \qquad \kappa=\mathrm{i}B\check{\gamma}_{r+1} \tag{+3i}$$

$$r=8k+5 \qquad \kappa=A\check{\gamma}_{r+1}\check{\gamma}_{r+2}. \tag{+4}$$

4.2. *Hyperbolic theories with r = 0*

By cases 3.3 and 3.6, and especially (20), we have

$$s=8k \qquad \text{or} \qquad 8k+4 \qquad \kappa=B. \tag{+i}$$

We exclude the case $s=8.0=0$. The cases are hyperbolic in the same sense as in case 4.1. For $k=0$ in the second case of (+i) we recognise the five-dimensional hyperbolic Kałuża-Klein theory with $(n,r,s)=(8,0,4)$. Similarly, by cases 3.3 and 3.5 we get

$$s=8k+7 \qquad \kappa=\mathrm{i}B\check{\gamma}_{s+1} \tag{+2i}$$

$$s=8k+7 \qquad \kappa=\mathrm{i}A\check{\gamma}_{s+1} \tag{+3}$$

$$s=8k+6 \qquad \kappa=B\hat{\gamma}_{s+1}\hat{\gamma}_{s+2}. \tag{+4i}$$

4.3. Symplectic theories with $s = 1$

By cases 3.1 and 3.6 we have

$$r = 8k+1 \qquad \text{or} \qquad 8k+5 \qquad \kappa = A. \tag{-1}$$

The cases are *symplectic* in the sense that, by choosing a suitable basis (e_j) of V, in each case the metric κ of V can be chosen as

$$\kappa = I_n := \begin{bmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{bmatrix}. \tag{26}$$

Similarly, by cases 3.3 and 3.5, especially (19), we get

$$r = 8k+2 \qquad \text{or} \qquad 8k+3 \qquad \kappa = \mathrm{i}B\hat{\gamma}_{s+1}. \tag{$-2\mathrm{i}$}$$

For $k = 0$ in the second case of $(-2\mathrm{i})$ we arrive at the five-dimensional symplectic Kałuża-Klein theory with $(n, r, s) = (8, 3, 1)$. Finally, by cases 3.3 we have

$$r = 4k+2 \qquad \kappa = \mathrm{i}A\hat{\gamma}_{s+1}. \tag{-3}$$

4.4. Symplectic theories with $r = 0$

By cases 3.2 and 3.5 we have

$$s = 8k+2 \qquad \text{or} \qquad 8k+6 \qquad \kappa = B. \tag{$-\mathrm{i}$}$$

The cases are symplectic in the same sense as in case 4.5. Similarly, by cases 3.4 and 3.6, and especially (20), we get

$$s = 8k+3 \qquad \text{or} \qquad 8k+4 \qquad \kappa = \mathrm{i}A\check{\gamma}_{r+1}. \tag{-2}$$

For $k = 0$ in the second case of (-2) we recognise the five-dimensional symplectic Kałuża-Klein theory with $(n, r, s) = (8, 0, 4)$. Finally, by case 3.4 we have

$$s = 8k+3 \qquad \kappa = \mathrm{i}B\check{\gamma}_{r+1}. \tag{$-3\mathrm{i}$}$$

All the distinguished cases are shown in figure 2. The choice of the index function of the Kałuża-Klein theories with values $m\delta$ and $m\delta i$, $m = 1, 2, 3, 4$; $\delta = 1, -1$ will be fully described in a separate paper (Ławrynowicz *et al* 1988).

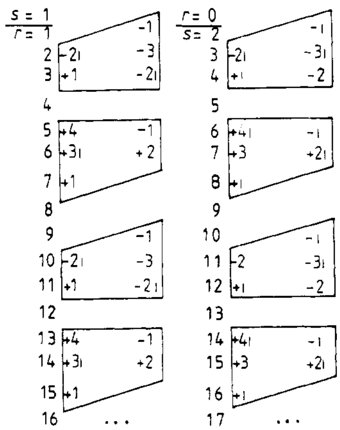

Figure 2. The Kałuża-Klein theories.

5. The concept of curved pseudo-Euclidean Hurwitz pairs and the corresponding generalised Fueter equations

In order to have a full generality of the Kałuża-Klein theories, according to the postulates (i)-(v) of § 3, we need the concept of a curved pseudo-Euclidean Hurwitz pair or, rather, a pseudo-Riemannian Hurwitz pair. In general, one of the ways to realise this idea is to apply the moving frames formalism (cf e.g. Sternberg 1964, pp 244-51). Because of our choice of a four-dimensional base space $\mathbb{M}_0$ in the postulate (iii), it is especially convenient to work with a particular case of that formalism—the tetrad formalism (cf e.g. Hehl and Datta 1971). Thus, if $z=(z^a)$ and $\zeta(z)=(\zeta^\alpha)(z)$ are local coordinates in $\mathbb{M}$ around z_0 and on the tangent space to $\mathbb{M}$ at z_0 (the latter coordinates being interpreted as the inertial ones), then the *field of tetrads* λ can locally be expressed by relations

$$(\partial/\partial\zeta^\alpha)\lambda^\alpha_a=\partial/\partial z^a. \tag{27}$$

The pseudo-Riemannian tensor of $\boldsymbol{B}_{\mathbb{M}}$, i.e. the pseudo-Riemannian metric of $\mathbb{M}$, is locally given by $g_{ab}=\lambda^\alpha_a\lambda^\beta_b\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the metric of the tangent space. Hence the formula (24) can be written, in the tetrad field notation, as

$$[\lambda^\alpha_a(z)]=\begin{bmatrix}[\lambda^m_\mu(x)] & [A^\beta_\mu(x)K^j_\beta(y)]\\ 0^T & [\lambda^\iota_k(y)]\end{bmatrix} \tag{28}$$

0 being the zero $4\times(p-4)$ matrix.

Now, consider the pair $(\boldsymbol{B}_{\mathbb{M}}, V)$ such that at every point z of $\mathbb{M}$ the tangent space to $\mathbb{M}$ at z forms, together with V, a pseudo-Euclidean Hurwitz pair. Then $(\boldsymbol{B}_{\mathbb{M}}, V)$ is called a *pseudo-Riemannian Hurwitz pair* with metric (24) or, equivalently, with the field of tetrads (28). In terms of tetrads the multiplication scheme (4) for base vectors (ε_α) and (e_j) has to be replaced by

$$\varepsilon_a(z)e_k=c^k_{ja}(z)e_k \qquad a=1,\ldots,p \qquad j,k=1,\ldots,n \tag{29}$$

where

$$c^k_{ja}(z)=\lambda^\alpha_a(z)c^k_{j\alpha}$$

and the basic formula (5) has to be replaced by

$$C_a(z)\bar{C}_b(z)+C_b(z)\bar{C}_a(z)=2g_{ab}(z)I_n \qquad a,b=1,\ldots,p \tag{30}$$

where

$$C_a:=[c^k_{j\alpha}] \qquad \bar{C}_a:=\kappa C^T_a\kappa^{-1}.$$

The concept of $(\boldsymbol{B}_{\mathbb{M}}, V)$ can still be generalised by replacing V with a pseudo-Riemannian or symplectic manifold $\mathbb{V}$ whose tangent bundle consists of the spaces meant in the previous sense. The main motivation for such a generalisation is given in the theorem of Gaveau *et al* (1982, 1985). If the principal fibre bundle $P(\mathbb{V}, G)$, where $\mathbb{V}$ is the base space and G is a semi-simple Lie group, is not trivial and admits solenoidal connection, then $\mathbb{V}$ is multiply connected. The theorem motivates in an elegant way the assumption of multiple connectivity made in earlier papers, e.g. by Misner and Wheeler (1957), Dirac (1964), Sakharov (1972), Ławrynowicz and Wojtczak (1974, 1977) and Ławrynowicz (1982); cf also Henkin (1981).

Let us consider now a continuously differentiable V-valued mapping f with a domain in $\mathbb{M}$ and the related spinor

$$\Psi = \kappa(f^1, \ldots, f^n)^T \tag{31}$$

where $n = \dim V$. Then it seems natural, following theorem 3 of our preceding paper (Ławrynowicz and Rembieliński 1986), to define the *generalised Fueter equation* (an analogue of the Cauchy-Riemann equations) as

$$[\Sigma_{\alpha \neq r+1}(-\mathrm{i}\gamma_\alpha \nabla^\alpha) + I_n \nabla^{r+1}]\Psi = 0 \tag{32}$$

and to call the mapping f, related to any solution of (32) given by (31), a *regular mapping*. Here ∇^α, $\alpha = 1, \ldots, p$, $\alpha \neq p$ ($p = r+s+1 = \dim \mathbb{M}$), and ∇^{r+1} are the covariant derivative analogues of $\partial^\alpha = \partial/\partial z^\alpha$ known from the flat case. We are going to calculate these operators effectively.

Consider arbitrary rotations of the local coordinates $\zeta(z)$ around z_0 in the tangent space to $\mathbb{M}$ at z_0: $\zeta(z) \mapsto \zeta'(z)$. If we define Λ by $\zeta'(z) = \Lambda(z)\zeta(z)$, then the field of tetrads (27) is transformed according to the formula

$$\lambda'^{\alpha}_{a} = \Lambda^{\alpha}_{\beta}\lambda^{\beta}_{a} \tag{33}$$

where $[\Lambda^\alpha_\beta] = \Lambda$. Then, for any representation $D \equiv D[\Lambda(z)]$, we have

$$\nabla_\alpha = \lambda^a_\alpha[(\partial/\partial z^a) + \Gamma_a] \tag{34}$$

where, according to (27), $(\partial/\partial z^a)\lambda^a_\alpha = \partial/\partial\zeta^a$ and Γ_a is the affine connection. Under the transformation $\zeta(z) \mapsto \zeta'(z)$ the connection Γ_a is transformed according to the standard formula

$$\Gamma'_a = D[\Lambda]\Gamma_a D^{-1}[\Lambda] + D[\Lambda](\partial/\partial z^a) D^{-1}[\Lambda]. \tag{35}$$

In order to express Γ_a by the tetrads, we denote by $\Sigma_{\alpha\beta}$ the generators of the semi-simple Lie group $\mathrm{SO}(r+1, s)$ in the representation $D[\Lambda]$:

$$\begin{aligned} &\Sigma_{\alpha\beta} = (1/4\mathrm{i})[\gamma_\alpha, \gamma_\beta] \qquad \text{for} \qquad \alpha, \beta \neq r+1 \\ &\Sigma_{\alpha,r+1} = -\Sigma_{r+1,\alpha} = \gamma_\alpha. \end{aligned} \tag{36}$$

Since, by (33)-(35),

$$D[I_n + \delta\Lambda] = I_n + \tfrac{1}{2}\omega^{\alpha\beta}\Sigma_{\alpha\beta} \tag{37}$$

where $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$, $\omega^{\alpha\beta}$ being real parameters, we get

$$\Gamma_a = \tfrac{1}{2}\Sigma^{\alpha\beta}\lambda^b_a\lambda^b_{\beta b;a}$$

where

$$\lambda_{\beta b;a} := \partial_a\lambda_{\beta b} - \tfrac{1}{2}\lambda^c_\beta(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ac})$$

with an obvious meaning of $\lambda_{\beta b}$. Equivalently, with the notation

$$\Omega_{[\alpha\beta]\gamma} := \lambda^a_\alpha\lambda^b_\beta(\partial_a\lambda_{\gamma b} - \partial_b\lambda_{\gamma a})$$

and

$$\Gamma_{a[\alpha\beta]} := \lambda^\gamma_a[\Omega_{[\gamma\alpha]\beta} - \Omega_{[\alpha\beta]\gamma} + \Omega_{[\beta\gamma]\alpha}]$$

we get

$$\Gamma_a = \tfrac{1}{2}\Sigma^{\alpha\beta}\Gamma_{a[\alpha\beta]}. \tag{38}$$

Turning our attention to ∇^{α}, by (34), we obtain

$$\nabla_{\alpha} = \lambda^{a}_{\alpha}\partial_{a} + \tfrac{1}{2}\mathrm{i}\Sigma^{\beta\gamma}\lambda^{b}_{\beta}\lambda^{a}_{\alpha}\lambda_{\gamma b;a}.$$

Now we can easily pass to ∇^{α}:

$$\nabla^{\alpha} = \lambda^{\alpha a}\partial_{a} + \tfrac{1}{2}\mathrm{i}\Sigma_{\beta\gamma}\lambda^{\beta b}\lambda^{\alpha a}\lambda^{\gamma}_{b;a} \tag{39}$$

with an obvious meaning of $\lambda^{\alpha a}$ and $\lambda^{\gamma}_{b;a}$. Therefore, by (37) and (38) we have

$$D[\Lambda] = \exp(\tfrac{1}{2}\mathrm{i}\omega^{\alpha\beta}\Sigma_{\alpha\beta})\exp(\mathrm{i}v^{\alpha}\gamma_{\alpha}) \tag{40}$$

$$\bar{D}[\Lambda] = \exp(\mathrm{i}v^{\alpha}\gamma_{a})\exp(-\tfrac{1}{2}\mathrm{i}\omega^{\alpha\beta}\Sigma_{\alpha\beta}) \tag{41}$$

where v^{α} are real parameters and, by (32),

$$-\mathrm{i}\gamma_{\alpha}\partial'^{\alpha} + I_{n}\partial'^{r+1} = \bar{D}[\Lambda](-\mathrm{i}\gamma_{\alpha}\partial^{\alpha} + I_{n}\partial^{r+1})D[\Lambda]. \tag{42}$$

Formulae (39)-(42) suffice for studying the symmetries in the Kałuża-Klein theories yielded by the generalised Fueter equations of the type (32).

The *action integral* corresponding to the matter, gauge and gravitational fields has to consist of two addends: the matter part I_{matt} and the gravitational part I_{grav}. We consider functionals of the form

$$I_{\mathrm{matt}} = \int \mathrm{d}^{4}z \, \det[\lambda^{\alpha a}]L_{\mathrm{matt}}[\Psi, \nabla_{\alpha}\Psi]$$

and

$$I_{\mathrm{grav}} = -(1/16\pi)G\int \mathrm{d}^{4}z \, \det[\lambda^{\alpha a}]R$$

where G is the gravitational constant and R is the scalar curvature on $\mathbb{M}$. The Dirichlet-like equations of motion (32) can be derived from the variational principle for $I_{\mathrm{matt}} + I_{\mathrm{grav}}$, where the variations have to be taken with respect to the spinors Ψ and tetrads λ. The formulae obtained are the starting point for quantisation according to one of known methods, e.g. the harmonic expansion method and quantisation of the zero modes; cf Mecklenburg (1984) and Strathdee (1986).

6. Symmetries in the Kałuża–Klein theories yielded by the generalised Fueter equations

Formulae (39)-(42) show that the internal symmetries in the Kałuża-Klein theories are described by the structural groups $\mathrm{SO}(r+1, s)$ whose generators $\Sigma_{\alpha\beta}$ have been used in those formulae. Besides, by the same formulae, the invariance group of any generalised Fueter equation (32) is $\mathrm{O}(r+1, s) \otimes T^{r+s+1}$, where $\mathrm{O}(r+1, s)$ is the group of pseudorotations in $T_{z}\mathbb{M}$ including $\mathrm{SO}(R+1, s)$ as its subgroup, T^{r+s+1} is the corresponding group of translations, and $\otimes$ denotes their semi-direct multiplication. Explicitly, we have

$$z' = \Lambda z + \boldsymbol{a} \qquad \Lambda \in \mathrm{O}(r+1, s)$$

where

$$\Psi'(z') = D[\Lambda]\Psi(z) \qquad \boldsymbol{a} \in T^{r+s+1}.$$

In particular we may take

$$\Lambda = \begin{bmatrix} \lambda & 0^{T} \\ 0 & z_{2} \end{bmatrix}$$

$\lambda \in \mathrm{O}(r, s)$, not including the direction $r+1$, $z_2 \in Z_2$, the cyclic two elements-group, 0 being the zero $1\times(r+s)$ matrix, so that D is the spinor representation of the group $Z_2 \times \mathrm{Pin}(r, s)$.

The last statement enables us to give a new interpretation of the generalised Fueter equations (32). Replacing $\mathrm{O}(r, s)$ by its subgroup $\mathrm{SO}(r, s)$ and defining

(a) the SO *ellipticity* of (32) as corresponding to $\mathrm{SO}(s) = \mathrm{SO}(0, s)$ (independent of whether $\kappa^T = \kappa$ or $\kappa^T = -\kappa$),

(b) the SO *hyperbolicity* of (32) as corresponding to $\mathrm{SO}(1, s)$,

we arrive at a duality of SO-hyperbolic and SO-elliptic Dirac-like equations (32) in the following cases:

$$\begin{array}{llll} s=1 & r=8k+3 & r=0 & s=8k+4 \\ & \text{case } (+1) \leftrightarrow \text{case } (-2) & & \\ & \text{case } (-2\mathrm{i}) \leftrightarrow \text{case } (+\mathrm{i}) & & \end{array} \tag{43}$$

$$\begin{array}{llll} s=1 & r=8k+5 & r=0 & s=8k+6 \\ & \text{case } (+4) \leftrightarrow \text{case } (-\mathrm{i}) & & \\ & \text{case } (-1) \leftrightarrow \text{case } (+4\mathrm{i}). & & \end{array} \tag{44}$$

Then, for the sake of simplicity, we assume that in the cases $(-2\mathrm{i})$ and (-2) in (43) with the same k we have the same second member $\mathbb{V} = \mathbb{V}_k$ in the Hurwitz pairs in question and consider their duality:

$$\begin{array}{llll} s=1 & r=8k+3 & r=0 & s=8k+4 \\ \text{case } (+1) \leftrightarrow \text{case } (-2) & & & \\ \text{case } (-2\mathrm{i}) \leftrightarrow \text{case } (+\mathrm{i}). & & & \end{array} \tag{45}$$

Similarly, we assume that in the cases (-1) and $(-\mathrm{i})$ in (44) with the same k we have the same second member $\mathbb{V} = \mathbb{V}'_k$ in the Hurwitz pairs concerned and consider their duality:

$$\begin{array}{llll} s=1 & r=8k+5 & r=0 & s=8k+6 \\ & \text{case } (+4) \leftrightarrow \text{case } (-\mathrm{i}) & & \\ & \text{case } (-1) \leftrightarrow \text{case } (+4\mathrm{i}). & & \end{array} \tag{46}$$

In order to better understand these dualities we have to observe that up to this section we have taken into account only the symmetry between the Hurwitz pairs corresponding to $(r+1, s)$ and $(s+1, r)$, expressed precisely by the formulae (13) and (14) and figure 2. In this section we are taking into account not only that symmetry, but also another, namely one between the Hurwitz pairs corresponding to (r, s) and (s, r), expressed precisely by the formulae (11) and (12).

If we wished to extend this procedure for the other cases appearing in figure 2, we should go outside the class of Kałuża-Klein theories.

For example, we concentrate on the cases (45) with $k=0$, i.e. the cases (22), and put

$$\boldsymbol{x} \equiv (x^1, x^2, x^3) \qquad \text{and} \qquad \tau \equiv x^4$$

for $\mathbb{M}_0$ consisting of all $(\boldsymbol{x}, \tau)$,

$$x_0 \equiv x^5$$

for $\mathbb{M}_{\#}$ consisting of all x^0. According to the considerations of cases 3.5 and 3.6, we are taking in V, $\dim V = 8$, the purely imaginary Majorana representation and the eight-dimensional spinors Ψ in equation (32).

Let us denote by $\tilde{\gamma}_\mu$, $\mu=0,\ldots,3$, the Clifford matrices corresponding to the representation in question. Then $\tilde{\gamma}_\mu$ can be expressed by the usual four-dimensional Majorana representation γ_μ, $\mu=0,\ldots,3$, as follows:

$$\tilde{\gamma}_\mu=\begin{bmatrix} 0_4 & \gamma_\mu \\ -\gamma_\mu & 0_4 \end{bmatrix} \qquad \gamma_0=\mathrm{i}\begin{bmatrix} 0_2 & \theta_0 \\ -\theta_0 & 0_2 \end{bmatrix} \qquad \gamma_1=\mathrm{i}\begin{bmatrix} 0_2 & \theta_1 \\ \theta_1 & 0_2 \end{bmatrix}$$

$$\gamma_2=\mathrm{i}\begin{bmatrix} -\theta_0 & 0_2 \\ 0_2 & \theta_0 \end{bmatrix} \qquad \gamma_3=\mathrm{i}\begin{bmatrix} 0_2 & \theta_3 \\ \theta_3 & 0_2 \end{bmatrix}$$

0_m being the zero $m\times m$ matrix;

$$\theta_0=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \theta_1=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \theta_3=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices $\tilde{\gamma}_\mu$ can be 'diagonalised' with the help of the following transformation $\boldsymbol{S}$:

$$\boldsymbol{S}\gamma_\mu\boldsymbol{S}^{-1}=-\mathrm{i}\begin{bmatrix} \gamma_\mu & 0_4 \\ 0_4 & -\gamma_\mu \end{bmatrix} \qquad \boldsymbol{S}=2^{-1/2}\begin{bmatrix} I_4 & \mathrm{i}I_4 \\ \mathrm{i}I_4 & I_4 \end{bmatrix}. \tag{47}$$

Hence the transformations shown in (45) by arrows are fully determined by the transformation $\boldsymbol{S}$ in (47).

Thus we may confine ourselves to the symplectic cases $(-2\mathrm{i})$ and (-2). We distinguish in Ψ four-dimensional spinors Ψ_+ and Ψ_-:

$$\Psi=\begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix} \qquad \boldsymbol{S}\begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix}=\begin{bmatrix} \Psi_D \\ \bar{\Psi}_D \end{bmatrix}$$

where

$$\Psi_\mathrm{D}=2^{-1/2}(\Psi_+ + \mathrm{i}\Psi_-).$$

Then in both cases the Fueter equation (32) becomes

$$(-\gamma_\mu\nabla^\mu + I_4\nabla^\mu)\Psi_\mathrm{D}=0 \tag{48}$$

so that Ψ_D has to be interpreted as the *Dirac field.* In such a way we get an additional motivation for calling the Fueter equation (32) a Dirac-like equation as we already did.

In the simplest case of $\mathbb{M}=M_4\times S_1$, mentioned in the introduction, independently of whether the case is hyperbolic: $(+1)$, $(+\mathrm{i})$ or symplectic: $(-2\mathrm{i})$, (-2), the vacuum solution (23) in the postulate (iii) of §4 has the form

$$[g^0_{ab}]=\begin{bmatrix} [\eta_{\mu\nu}] & 0 \\ 0^T & 1 \end{bmatrix}$$

0 being the zero 1×4 matrix, where $[\eta_{ab}]$ is the usual Minkowski metric or the anti-Minkowski metric. Then the tensor (24) in the Kałuża–Klein ansatz takes the form

$$[g_{ab}]=\begin{bmatrix} [\eta_{\mu\nu}+e^2A_\mu A_\nu] & [-eA_\mu] \\ [-eA_\nu] & 1 \end{bmatrix}. \tag{49}$$

The corresponding tetrad field (28) becomes

$$[\lambda^\alpha_a]=\begin{bmatrix} [\delta^\nu_\mu] & [eA_\mu] \\ 0^T & 1 \end{bmatrix}$$

0 being the zero 1×4 matrix.

If we denote by R the radius of S^1, then the metric (49) is invariant under any transformation of the form

$$x' = x \qquad \tau' = \tau + R\phi(x) \qquad A'_\mu = A_\mu - (R/e)\partial_\mu \phi(x).$$

In the cases (−2i) and (−2) the Fueter equation (48) becomes

$$\{-\gamma_\mu[\partial^\mu - \mathrm{i}(e/R)A^\mu] + \partial^\tau I_4\}\Psi_D(x, \tau) = 0$$

and we also have the transversality condition $\partial^\tau \Psi_D = (\mathrm{i}/R)\Psi_D$. Hence

$$\Psi_D(x, \tau) = \Psi_D(x) \exp[(\mathrm{i}/R)\tau].$$

Finally we obtain

$$\{-\gamma_\mu[\partial^\mu - \mathrm{i}(e/R)A^\mu] + (\mathrm{i}/R)I_4\}\Psi_D(x) = 0. \tag{50}$$

We conclude that in the symplectic cases (−2i) and (−2) the Fueter equation (50) describes the Dirac field of mass $\sim 1/R$, interacting with the electromagnetic field.

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